

CHIRAL SYMMETRY OUTSIDE PERTURBATION THEORY¹

H. NEUBERGER

*Department of Physics and Astronomy
Rutgers University
Piscataway, NJ 08855
U.S.A.*

Abstract. Within the overlap framework, I derive the main formulae one finds today in papers touting a “new approach” to the regularization of chiral gauge theories. My main objective is to clear up an unhealthy confusion about how many successful approaches to regulate chiral fermions on the lattice there really are: At the moment, there is only one, the overlap, and finding a genuinely different approach is an important and completely open problem.

1. Introduction

The talk I delivered at this workshop had substantial overlap with talks I gave at Lattice’99 [1] and at Chiral’99 [2]. To avoid repetition, this write-up is restricted to technical points which were neither covered orally, nor in the above mentioned written contributions.

I shall show explicitly how the main formulae one finds today in papers touting a “new approach” to the regularization of chiral gauge theories [3, 4, 5, 6] are directly and straightforwardly derived from the overlap. Thus, there really is no “new” approach and there are fewer new results than a superficial reading of the above papers would indicate. Whether one likes to start from the Ginsparg-Wilson [7] relation or from the overlap one arrives at the same algebraic setup. The GW relation, by itself, does not guarantee the right dynamics, and one needs to fulfill extra side conditions in order to really get chiral fermions [2]. The GW relation became fashionable in January 1998. The overlap Dirac operator first appeared in [8] (posted July, 1997) and its connection to the GW relation was pointed out in [9] (posted

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October, 1997). This write-up consists of a collection of formulae related to overlap chiral fermions on the lattice with their derivations; each derivation amounts to a little exercise in linear algebra and is quite trivial.

My main objective in putting this summary in print has been stated in the abstract. To be sure, let me add that *some* new results have indeed been obtained recently: During the last two years a certain amount of progress has taken place on the mathematical question of fine tuning the phase of the overlap to eliminate small gauge breaking effects when anomalies cancel.

2. Notation

Our focus is on lattice Dirac fermions defined on a four dimensional hyper-cubic lattice in a background of $SU(n)$ lattice gauge fields. These fermions live in a finite complex vector space of even dimension N . Elements in this space will be denoted as \vec{v} . Components of these vectors will be labeled by combined indices, I, J , with the convention $I = (x, i, \alpha)$, $J = (y, j, \beta)$, where x, y label sites, i, j group indices and α, β spinor indices. Operators will be represented by matrices with matrix indices appearing as subscripts. If only site indices appear the group and spinor indices are to be understood as suppressed. Trace operations either operate on all indices (Tr) or only on a restricted set, typically excluding sites (tr). In addition to square matrices we shall often employ rectangular ones, dimensioned as (number of rows) \times (number of columns).

Two main reflection operators act on the vectors \vec{v} : ϵ and ϵ' . A reflection is a unitary-hermitian operator. Equivalently, one can think about the associated projectors $P = \frac{1}{2}(1 - \epsilon)$, $P' = \frac{1}{2}(1 - \epsilon')$ as fundamental. The Kato [10] pair $h = \frac{1}{2}(\epsilon + \epsilon')$ and $s = \frac{1}{2}(\epsilon - \epsilon')$ is algebraically characterized [11] by $h^2 + s^2 = 1$ and $\{h, s\} = 0$, where the anticommutator can be viewed as a version of the Ginsparg-Wilson relation, or a lattice version of chiral symmetry [2]. A central role is played by the overlap Dirac operator [8] $D_o = \frac{1}{2}(1 + \epsilon'\epsilon) = \epsilon'h$. Nothing of principle is lost by setting $\epsilon' = \gamma_5$. Some trivial identities are listed below:

$$\epsilon'h = h\epsilon, P'h = hP, \epsilon h = h\epsilon', Ph = hP'. \quad (1)$$

In the above one can replace h by h^{-1} when the inverse exists. Similar equations, up to signs, are obeyed by s . Another trivial identity is $h = 1 - P - P'$. From it one derives $P'h^{-1}P' = -P'$ and similarly

$$Ph^{-1}P = -P, (1 - P')h^{-1}(1 - P') = 1 - P', (1 - P)h^{-1}(1 - P) = 1 - P. \quad (2)$$

The inverse of the overlap Dirac operator obeys:

$$D_o^{-1} - 1 = \frac{1 - \epsilon'\epsilon}{1 + \epsilon'\epsilon} = sh^{-1} = -h^{-1}s. \quad (3)$$

The last expressions obviously anti-commute with ϵ' and are anti-hermitian.

A second quantized notation will also be employed when appropriate: We imagine dealing with a system of N noninteracting fermions represented by a 2^N dimensional Fock space. The elements making up the Fock space are superpositions of anti-symmetrized direct products of single particle states obtained using any basis of the original complex N -dimensional space one chooses. One assumes a standard basis relatively to which standard fermionic creation/annihilation operators a_I^\dagger/a_I are defined. To distinguish vectors in the Fock space from vectors in other spaces we shall use Dirac bra-ket notation for second quantized states only.

In the overlap one needs to fill all the negative energy states of ϵ . They generate $\mathbf{Ker}(1 + \epsilon) = \mathbf{span}\{\vec{v}_i | i = 1, 2, \dots, N_v\}$. ϵ depends on the gauge fields $U_\mu(x)$ and so do the orthonormal vectors \vec{v}_i . Similarly one introduces $\mathbf{Ker}(1 - \epsilon) = \mathbf{span}\{\vec{w}_i | i = 1, 2, \dots, N_w\}$, with $N_v + N_w = N$. Appending a prime, similar objects are introduced for ϵ' , but now there is no gauge field dependence and $N_{v'} = N_{w'} = \frac{1}{2}N$. The Dirac sea state corresponding to occupying all \vec{v}_i single fermion states will be denoted by $|v\rangle$.

It is convenient to collect all the vectors \vec{v}_i into an $N \times N_v$ matrix $v = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{N_v})$ and do the same for similar collections of vectors. Then:

$$P = vv^\dagger, \quad v^\dagger v = 1, \quad 1 - P = ww^\dagger, \quad w^\dagger w = 1, \quad (4)$$

$$Pv = v, \quad P'v' = v', \quad Pv' = D_o^\dagger v', \quad P'v = D_o v. \quad (5)$$

Starting from identities like $v'^\dagger h^{-1} v' = v'^\dagger P' h^{-1} P' v'$ we get:

$$v'^\dagger h^{-1} v' = v^\dagger h^{-1} v = -1, \quad w'^\dagger h^{-1} w' = w^\dagger h^{-1} w = 1. \quad (6)$$

3. Overlap determinant and propagator

3.1. CHIRAL CASE

Pauli's statistics for fermions implies

$$\langle v' | v \rangle = \det M_R, \quad M_R = v'^\dagger v. \quad (7)$$

This is the overlap and gives the lattice chiral fermion determinant. By convention, it is associated with right handed Weyl fermions (R). The inverse of M_R is

$$M_R^{-1} = -v^\dagger h^{-1} v' \quad (8)$$

because $-v'^\dagger v v^\dagger h^{-1} v' = -v'^\dagger P h^{-1} v' = -v'^\dagger h^{-1} v' = 1$.

The matrix M_R can be rewritten in an artificial way to look dependent also explicitly on the overlap Dirac operator:

$$M_R = v'^\dagger D_o v. \quad (9)$$

The equation holds by taking the factors ϵ' and ϵ in D_o to act left and right respectively. This is equation (18) in [3]. It is strange that the simpler overlap form is never even mentioned in [3, 4, 5, 6].

M_R^{-1} is not directly equal to the R-fermion propagator, G^R . Unlike M_R , G^R is an operator on the original vector space. Since it represents the propagation of R -Weyl fermions but acts in a space that accommodates Dirac fermions it is appropriately rank deficient. In the overlap construction (equation (5.19) in [12]) the propagator is found to be

$$G_{JI}^R = \frac{\langle v' | a_I^\dagger a_J | v \rangle}{\langle v' | v \rangle}, \quad G^R = v M_R^{-1} v'^\dagger. \quad (10)$$

When it exists, G^R has rank $\frac{N}{2}$ and is given by $G^R = -P h^{-1} P' = -h^{-1} P'$. Since $\epsilon' P' = -P'$

$$G^R = P h^{-1} \epsilon' P' = P D_o^{-1} P'. \quad (11)$$

The last expression is equation (17) in [3], but $G^R = -h^{-1} P'$ is simpler.

The formula $G^R = -h^{-1} P'$ is important because it makes it explicit that G^R transforms covariantly under gauge transformations. This covariance is self-evident in the second quantized expression in terms of operators. Thus, $\text{tr} \Gamma G_{x,y}^R W_{y,x}$, where $W_{y,x}$ is a Wilson line operator connecting sites x and y and Γ acts only on spinorial indices, is gauge invariant although $\det M_R$ is generically not. In this aspect the overlap is different from earlier attempts to put chiral fermions on the lattice, break gauge invariance and restore it subsequently by gauge averaging. In the overlap gauge averaging cannot destroy the perturbative masslessness of the fermions, so, for instance, the counter example of Testa [13] does not apply.

For L -fermions we introduce $M_L = w'^\dagger w$ with inverse $M_L^{-1} = w^\dagger h^{-1} w'$ and propagator $G^L = h^{-1}(1 - P') = (1 - P)h^{-1}$, obtained by reversing the signs of ϵ and ϵ' [12]. This leads to

$$G^R + G^{L\dagger} = (1 - P - P')h^{-1} = 1. \quad (12)$$

To better match continuum properties [12] we define the overlap chiral external propagators $G_o^R = G^R - \frac{1}{2}$ and $G_o^L = G^L - \frac{1}{2}$. This corresponds to replacing $a_I^\dagger a_J$ by $\frac{1}{2}(a_I^\dagger a_J - a_J a_I^\dagger)$ in the second quantized formula (equation (5.22) in [12]). One can take an R - L combination to propagate with $G_o^R + G_o^L \equiv G^V$.

$$G^V = -h^{-1} P' + h^{-1}(1 - P') - 1 = h^{-1} \epsilon' - 1 = D_o^{-1} - 1 = s h^{-1} \quad (13)$$

anti-commutes with ϵ' ; however, $G^R + G^L = D_o^{-1}$ does not.

3.2. VECTOR-LIKE CASE

The fermion determinant in the vector-like case is $|\det M_R|^2 = \det v'^\dagger P v'$. Hence $|\det M_R|^2 =$

$$\det[1 - v'^\dagger(1 - P)v'] = e^{\sum_{m=1}^{\infty} \frac{1}{m} \text{Tr} [P'(1-P)]^m} = \det[1 - P'(1 - P)]. \quad (14)$$

Similarly, one shows $|\det M_R|^2 = |\det M_L|^2 =$

$$\det[1 - (1 - P)P'] = \det[1 - P(1 - P')] = \det[1 - (1 - P')P]. \quad (15)$$

Comparing the matrix elements between the bases $\{\vec{v}'_i, \vec{w}'_j\}$ and $\{\vec{v}_i, \vec{w}_j\}$ of the above combinations of projectors with those of D_o we obtain

$$|\det M_R|^2 = |\det M_L|^2 = \det D_o. \quad (16)$$

Since $\det \epsilon' = 1$, $\det D_o = \det h$. The propagator on internal fermion lines is given by $D_o^{-1} = G^R + G^L$. Only on external fermion lines can one use $G^V = D_o^{-1} - 1$ and preserve naïve chiral symmetry exactly [14].

4. Consistent, Covariant Currents, Anomalies and Topology

To get currents one computes the first order variation of the chiral determinant with respect to the gauge fields. The precise form of the variation (δ) is not important here. The variation can be naturally (geometrically) decomposed into two terms [15, 16, 17].

$$\delta \log \langle v' | v \rangle = \frac{\langle v' | \delta v_\perp \rangle}{\langle v' | v \rangle} + \langle v | \delta v \rangle, \quad (17)$$

to isolate the dependence on the phase choice for $|v\rangle$ into the local last term. The main point is that because of the phase ambiguity the component of $|\delta v\rangle$ along $|v\rangle$ is not determined by $\delta\epsilon$ but $|\delta v_\perp\rangle$ is. The first term, being phase choice independent can be made to transform covariantly under gauge transformations with an appropriate choice of the variation δ . This term is nonlocal in gauge fields and defines the covariant current. The sum of both terms is also a current and, assuming a single valued (as a function of the gauge background) choice of the second quantized states $|v\rangle$ has been made, gives the consistent current with an appropriate choice of δ . The difference, denoted by ΔJ in the continuum [18], is the last term. This last term is recognized as the Berry connection [19]: Under a change of phase $|v\rangle \rightarrow e^{i\Phi(U)}|v\rangle$ it changes additively by $i\delta\Phi$, but it contains invariant information in associating a Berry phase $\Phi(C)$ with every closed contour C in gauge field space.

Let us now translate back to first quantized language:

$$\langle v|\delta v\rangle = \sum_{i=1}^{N_v} \vec{v}_i^\dagger \delta \vec{v}_i = \mathbf{Tr} \ v^\dagger \delta v. \quad (18)$$

For simplicity, we assume $\langle v'|v\rangle \neq 0$ (which also implies $N_v = \frac{N}{2}$).

The variation of $\log\langle v'|v\rangle$ is just $\mathbf{Tr} \ M^{-1}\delta M = -\mathbf{Tr} \ v^\dagger h^{-1} v' v'^\dagger \delta v = -\mathbf{Tr} \ v^\dagger h^{-1} \delta v = -\mathbf{Tr} \ v^\dagger h^{-1} \delta(Pv)$ which is equal to:

$$-\mathbf{Tr} \ h^{-1} P' \delta P - \mathbf{Tr} \ v^\dagger P h^{-1} P \delta v = \mathbf{Tr} \ v^\dagger \delta v + \mathbf{Tr} \ P' \delta h h^{-1} \quad (19)$$

The last term can be rewritten as

$$\mathbf{Tr} \ P' \delta h h^{-1} = -\mathbf{Tr} \ \epsilon' \delta h h^{-1} P' = \mathbf{Tr} \ \delta D_o G^R, \quad (20)$$

leading to an expression for the covariant current:

$$\frac{\langle v'|\delta v_\perp\rangle}{\langle v'|v\rangle} = \mathbf{Tr} \ P' \delta h h^{-1} = \mathbf{Tr} \ P' \delta D_o D_o^{-1} = \mathbf{Tr} \ \delta D_o G^R. \quad (21)$$

One should not forget however that G^R is not D_o^{-1} . The above equation contains formula (21) in [3]. In second quantized notation we have:

$$\frac{\langle v'|\delta v_\perp\rangle}{\langle v'|v\rangle} = \frac{\langle v'|a_I^\dagger (\delta D_o)_{IJ} a_J|v\rangle}{\langle v'|v\rangle}. \quad (22)$$

So, all we calculated is the bilinear numerical kernel giving the current operator associated with varying the fermion induced effective action in second quantized language. Actually, the second quantized form is advantageous in topologically nontrivial backgrounds. There, to make the expression meaningful one needs to insert some operator of the 't Hooft vertex type in the numerator (the denominator is “canceled” by the fermion determinant factor). Thus one can consider correlators between 't Hooft vertices and covariant currents.

To get the covariant anomaly we choose the variation to be an infinitesimal gauge transformation with parameters $\omega^a(x)$ at site x where a labels the $n^2 - 1$ hermitian generators t_r^a acting on the fermions which are in a representation (possibly reducible) r . The site diagonal matrix $\mathcal{G}_{I,J} = i\omega^a(x)\delta_{xy}(t_r^a)_{ij}\delta_{\alpha\beta}$ represents the transformation in fermion space. Thus, $\delta h = [\mathcal{G}, h]$, reflecting the covariance of h .

Starting from $\mathbf{Tr} \ [\mathcal{G}, h] h^{-1} P' =$

$$\mathbf{Tr} \ \mathcal{G} P' - \mathbf{Tr} \ h \mathcal{G} h^{-1} P' = \frac{1}{2} \mathbf{Tr} \ \mathcal{G} - \mathbf{Tr} \ h \mathcal{G} P h^{-1} = \frac{1}{2} \mathbf{Tr} \ \mathcal{G} \epsilon \quad (23)$$

we obtain for the covariant anomaly:

$$\frac{i}{2} \sum_x \omega^a(x) \mathbf{tr} (t_r^a \epsilon_{x,x}) \equiv i \sum_x \omega^a(x) \Delta^a(x). \quad (24)$$

In the equations above the tracelessness of ϵ' in spinor space was used. The anomaly $\Delta^a(x)$ can also be trivially rewritten as

$$\Delta^a(x) = \mathbf{tr} \epsilon' t_r^a (D_o)_{x,x}. \quad (25)$$

This is equivalent to equation (24) in [3] (there is a factor of two difference stemming from the choice in [3] to write the GW relation as $\{D, \gamma_5\} = D\gamma_5 D$ rather than the overlap form $\{D_o, \gamma_5\} = 2D_o\gamma_5 D_o$). The anomaly equation is meaningful even when D_o is not invertible.

The topological charge Q in the overlap has been long known [15, 20, 12] to be given by $\frac{1}{2} \mathbf{Tr} \epsilon$. Since

$$Q = \frac{1}{2} \mathbf{Tr} \epsilon = \frac{1}{2} \sum_x \mathbf{tr} \epsilon_{x,x} = \sum_x \Delta^{U(1)}(x), \quad (26)$$

we see the expected relation between the anomaly and the index.

5. Berry phase issues

5.1. BERRY'S CURVATURE

We already introduced Berry's connection $\mathcal{A}_i = \langle v | \delta_i v \rangle = \mathbf{Tr} v^\dagger \delta_i v$. Under a phase change $|v\rangle \rightarrow e^{i\Phi} |v\rangle$ $\mathcal{A}_i \rightarrow \mathcal{A}_i + i\delta_i \Phi$, but the associated (abelian) Berry curvature \mathcal{F}_{12} is unaffected [17]:

$$\mathcal{F}_{12} = \delta_1 \mathcal{A}_2 - (1 \leftrightarrow 2) = \mathbf{Tr} \delta_1 v^\dagger \delta_2 v - (1 \leftrightarrow 2). \quad (27)$$

The phase freedom of the second quantized state $|v\rangle$ amounts to an arbitrary *unitary* \mathcal{O} rotation among the first quantized states making up the Dirac sea. $\mathbf{Tr} v^\dagger \delta v \rightarrow$

$$\mathbf{Tr} \mathcal{O}^\dagger v^\dagger \delta(v\mathcal{O}) = \mathbf{Tr} v^\dagger \delta v + \mathbf{Tr} \mathcal{O}^\dagger \delta \mathcal{O} = \mathbf{Tr} v^\dagger \delta v + \delta \log \det \mathcal{O}. \quad (28)$$

Hence $e^{i\Phi} = \det \mathcal{O}$, and the rest of \mathcal{O} is irrelevant. The intrinsic meaning of Berry's curvature is made explicit by expressing it in terms of the projectors only. The relevant formula is well known [21] and has been used in [17]. Start from

$$\mathbf{Tr} P \delta_1 P \delta_2 P = \mathbf{Tr} v v^\dagger (\delta_1 v v^\dagger + v \delta_1 v^\dagger) (\delta_2 v v^\dagger + v \delta_2 v^\dagger) \quad (29)$$

and expand the right hand side into a sum of four terms. Three of them are symmetric in the 1,2 indices and the one which is not is $\mathbf{Tr} \delta_1 v^\dagger \delta_2 v$.

Observing that $\mathbf{Tr} \delta_1 P \delta_2 P$ is also symmetric we can replace every P by a term $-\frac{1}{2}\epsilon$, obtaining

$$\mathcal{F}_{12} = -\frac{1}{8} \mathbf{Tr} \epsilon [\delta_1 \epsilon, \delta_2 \epsilon]. \quad (30)$$

This formula, with projectors in the reflections' stead, is eq (3.21) in [5].

Let us now choose explicit formulae for the variations. We parameterize the group by real coordinates ξ^a , so there is one set of ξ 's for every link: $\delta X = \frac{\partial X}{\partial \xi^a(x, \mu)} \delta \xi^a(x, \mu)$, with the summation convention acting on a, x, μ . One would rather use vector fields with nicer transformation properties than those of $\frac{\partial}{\partial \xi^a(x, \mu)}$. Focusing for the moment on a single copy of the group we opt to use the globally defined left invariant vector fields $I_b = u_b^a(\xi) \frac{\partial}{\partial \xi^a}$. The $I_a(\xi)$ vector fields represent the Lie Algebra (with real structure constants f_{ab}^c) acting on the group manifold: $[I_a(\xi), I_b(\xi)] = f_{ab}^c I_c(\xi)$. The main point about the introduction of the real matrix $u(\xi)$ [22] is that, for an arbitrary group element parameterized by ξ , $g(\xi)$, we have:

$$u_b^a(\xi) \frac{\partial g(\xi)}{\partial \xi^a} = g(\xi) \left(\frac{\partial g(\xi')}{\partial \xi'^b} \right)_{\xi'=0}. \quad (31)$$

This proves that for any two fixed group elements g_1 and g_2

$$I_a(\xi)(g_1 g(\xi) g_2) = (g_1 g(\xi) g_2) (g_2^{-1} \left(\frac{\partial g(\xi')}{\partial \xi'^a} \right)_{\xi'=0} g_2) \quad (32)$$

showing that gauge transformations act linearly on the covariant currents defined below.

Let $\mathbf{E}_a(x, \mu) = I_a(\xi(x, \mu))$ with $[\mathbf{E}_a(x, \mu), \mathbf{E}_b(y, \nu)] = f_{ab}^c \delta_{xy} \delta_{\mu\nu} \mathbf{E}_c(x, \mu)$.

$$\mathcal{A}_a(x, \mu) = \langle v | \mathbf{E}_a(x, \mu) v \rangle = \mathbf{Tr} v^\dagger \mathbf{E}_a(x, \mu) v \quad (33)$$

$$J_a^{\text{cov}}(x, \mu) = \frac{\langle v' | [(\mathbf{E}_a(x, \mu) v)_{\perp}] \rangle}{\langle v' | v \rangle} = \mathbf{Tr} [\mathbf{E}_a(x, \mu) D_o] G^R. \quad (34)$$

The antisymmetric tensor over field space $\mathcal{F}_{ab}(x, \mu; y, \nu)$ is given by

$$\langle \mathbf{E}_a(x, \mu) v | \mathbf{E}_b(y, \nu) v \rangle = \mathbf{Tr} [\mathbf{E}_a(x, \mu) v^\dagger] [\mathbf{E}_b(y, \nu) v], \text{ antisymmetrized.} \quad (35)$$

$$\mathcal{F}_{ab}(x, \mu; y, \nu) = \mathbf{Tr} P [\mathbf{E}_a(x, \mu) P, \mathbf{E}_b(y, \nu) P] = \mathbf{E}_a(x, \mu) \mathcal{A}_b(y, \nu)$$

$$- \mathbf{E}_b(y, \nu) \mathcal{A}_a(x, \mu) - f_{ab}^c \delta_{xy} \delta_{\mu\nu} \mathcal{A}_c(x, \mu). \quad (36)$$

My notation and definitions [17] are more general than in [6], so that they apply to any Lie Group and any representation. In the $SU(n)$ case with matrices in the fundamental the above reduces to equation (8.1) in [6].

5.2. $Z(2)$ ANOMALY

The gauge group $SU(2)$ with fermions in non-integral representations is an interesting case because one can take the basic vector space over the reals [23]. This makes Berry's connection and curvature vanish; still the state $|v\rangle$ has a sign ambiguity - the $U(1)$ bundle of the complex case has been replaced by a $Z(2)$ bundle. If the $Z(2)$ bundle is twisted (like a Möbius strip), one cannot find a single valued smooth definition for the states $|v\rangle$. If one could, the signs of all $|v\rangle$'s would be determined by smoothness up to an irrelevant overall sign. Then, lattice gauge transformations, which can always be smoothly deformed to the identity, could not induce sign changes and the chiral determinant would be gauge invariant.

In the continuum Witten [24] has shown that a gauge invariant formulation for representations $\frac{1}{2}, \frac{5}{2}, \dots$ is impossible because the space of *gauge orbits* (unlike the space of gauge fields) is multiply connected: There are two classes of closed curves and the chiral determinant when taken round a curve in the nontrivial class can change sign, so is not single valued, making a gauge invariant formulation impossible. A nontrivial curve is obtained from an open path connecting a gauge field configuration to its gauge transform by a gauge transformation which cannot be smoothly deformed to unity, and subsequently going to orbit space.

Although Witten's anomaly is reproduced on the lattice [25], the realization cannot be by exactly the same mechanism as in the continuum because on the lattice the space of all gauge transformations over the torus is connected (any lattice gauge transformation can be smoothly deformed to unity), while in the continuum it is not and this is the heart of the matter [24]. The space of gauge orbits is multiply connected also on the lattice, but this time as a consequence of the space of gauge fields itself being already multiply connected (unlike in the continuum) as a result of a necessary gauge invariant excision of backgrounds where ϵ cannot be unambiguously defined [25].

5.3. ADIABATICS

It is well known that Berry's phase [19] is captured by adiabatic Hamiltonian evolution. The quantum mechanical adiabatic theorem states that in the adiabatic limit Hamiltonian evolution can be replaced by a geometric evolution with an operator introduced by Kato [26] over forty years ago. The geometric evolution amounts to parallel transport with Berry's connection [27]. Considering a path in gauge field space parameterized by $t_1 \leq t \leq t_2$ this parallel transport means

$$\langle v_{\text{ad}} | \dot{v}_{\text{ad}} \rangle = 0, \quad (37)$$

where we assumed $\langle v_{\text{ad}} | v_{\text{ad}} \rangle \equiv 1$ and the dot denotes a derivative with respect to t .

If we set $|v_{\text{ad}}(t_1)\rangle = |v(t_1)\rangle$, where the “time” argument identifies a gauge field configuration, adiabatic evolution means that at all t $|v_{\text{ad}}(t)\rangle$ will be equal to $|v(t)\rangle$ up to phase, the phase of $|v_{\text{ad}}(t)\rangle$ being fixed relatively to that of $|v(t)\rangle$ by the above law of parallel transport.

In first quantized language the phase arbitrariness of the states $|v\rangle$ becomes the arbitrariness of v under a unitary rotation by \mathcal{O} . If all states \vec{v}_i , \vec{w}_j are evolved adiabatically along our path they become $\vec{v}_{\text{ad},i}$, $\vec{w}_{\text{ad},j}$ with the associated matrices related by

$$v(t) = v_{\text{ad}}(t)\mathcal{O}_v(t), \quad w(t) = w_{\text{ad}}(t)\mathcal{O}_w(t), \quad (38)$$

with initial conditions $\mathcal{O}_v(t_1) = 1$ and $\mathcal{O}_w(t_1) = 1$. The \mathcal{O} matrices are uniquely defined as a function of t by the law of adiabatic transport:

$$v_{\text{ad}}^\dagger(t)\dot{v}_{\text{ad}}(t) = 0, \quad w_{\text{ad}}^\dagger(t)\dot{w}_{\text{ad}}(t) = 0. \quad (39)$$

The unitary transformation $K(t)$ producing the time evolution of the adiabatically evolved basis is given by

$$v_{\text{ad}}(t)v^\dagger(t_1) + w_{\text{ad}}(t)w^\dagger(t_1) = v(t)\mathcal{O}_v^\dagger(t)v^\dagger(t_1) + w(t)\mathcal{O}_w^\dagger(t)w^\dagger(t_1). \quad (40)$$

Kato defined $K(t)$ by the following first order differential equation

$$\dot{K} = [\dot{P}, P]K, \quad K(t_1) = 1. \quad (41)$$

The formula is proven starting from $P(t) = v_{\text{ad}}(t)v_{\text{ad}}^\dagger(t)$ and observing:

$$\dot{P}Pv_{\text{ad}} = \dot{v}_{\text{ad}}, \quad P\dot{P}v_{\text{ad}} = 0, \quad P\dot{P}w_{\text{ad}} = -\dot{w}_{\text{ad}}. \quad (42)$$

$|v_{\text{ad}}(t)\rangle = e^{i\Phi(t)}|v(t)\rangle$ and, as we saw before, $e^{i\Phi(t)} = \det \mathcal{O}_v(t)$. To express $e^{i\Phi(t)}$ in terms of $K(t)$ we isolate $\mathcal{O}_v^\dagger(t)$

$$v^\dagger(t_1)v(t)\mathcal{O}_v^\dagger(t) = v^\dagger(t_1)K(t)v(t_1), \quad (43)$$

which leads, after adding and subtracting $1 = v^\dagger(t_1)v(t_1)$ to

$$e^{-i\Phi(t)} = \frac{\det[1 - P(t_1) + P(t_1)K(t)]}{\det v^\dagger(t_1)v(t)}. \quad (44)$$

Since $v(t)$ is single valued, $v(t_1) = v(t)$ for a *closed path* C , making the denominator unity and producing equation (6.6) of [6].

Similarly, observing that $M_R(t)\mathcal{O}_v^\dagger(t)M_R^\dagger(t_1) = v'^\dagger P(t)K(t)P(t_1)v' = v'^\dagger D_o(t)K(t)D_o^\dagger(t_1)v'$, we get

$$\det M_R(t) (\det M_R(t_1))^* = \det[1 - P' + P'D_o(t)K(t)D_o^\dagger(t_1)] \det \mathcal{O}_v(t), \quad (45)$$

which is equation (31) in [3].

5.4. CONNECTION TO CONTINUUM (SIMPLIFIED)

As long as we focus only on one open path there is no fundamental distinction between $|v(t)\rangle$ and $|v_{\text{ad}}(t)\rangle$. But, when we recall the environment in which the path is embedded, it is clear that the states $|v(t)\rangle$ come from a single valued function on the entire space of gauge fields. The state $|v_{\text{ad}}(t)\rangle$ typically does not return to its initial value when transported round a *closed* path C in gauge field space. The extra phase it acquires is Berry's phase $e^{i\Phi(C)}$. If $e^{i\Phi(C)} = 1$ for any closed curve C in gauge field space, a natural global phase choice would be to parallel transport by Berry's connection from one fixed point in field space. Since the exact structure of the reflection ϵ is not important for the continuum limit, but does determine $\Phi(C)$, it is natural to search for a deformation of ϵ ("improvement") so that $e^{i\Phi(C)} = 1$ for all closed curves C . It can be shown that *it is impossible* to find such a deformation if continuum perturbative anomalies *do not cancel* [17]. It is conjectured that if anomalies cancel and the above obstruction is removed a deformed ϵ with $e^{i\Phi(C)} \equiv 1$ exists and would permit a gauge invariant phase choice by Berry's law of parallel transport [17] (for a recent review see [2]).

The above conjectures are compatible with continuum: There, non-abelian anomalies cancel iff one can make the conserved and covariant currents identical, in other words when ΔJ can be made to vanish. In the overlap framework the role of ΔJ is played by Berry's connection and this motivates the conjectures. When anomalies do not cancel, the inevitability of nontrivial Berry phases was established in [17] by exhibiting a class of backgrounds on a torus *in gauge orbit space* over which Berry's curvature generated a two form which was well defined but integrated to a non-zero integer. The result was in agreement with known continuum formulae for ΔJ [28]. In the continuum, once ΔJ is known (including normalization), the anomalies themselves (both consistent and covariant) are completely determined. The idea to add two continuous directions (in the case of [17] these were the two torus coordinates) to produce integrands that integrate to integers (which can be non-zero if anomalies do not cancel, as was found in [6]) is at the core of section 9 in [6].

6. Conclusions

All recent algebraic relations are identities directly obtainable from the basic overlap formulae. The one new idea is an approach to look for a gauge variant local functional to be added to the phase of an assumed given smooth section of the initial $U(1)$ overlap bundle so that gauge invariance

holds when anomalies cancel. The new approach is an alternative to the older proposal of [17] and mathematically differs in replacing the geometrical framework of [17] by one based on analysis. Although both approaches include some fine tuning, in [17] the fine tuning is at the level of gauge covariant operators only. Space limitations prevent further discussion.

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References

1. H. Neuberger, hep-lat/9909042.
2. H. Neuberger, hep-lat/9911022.
3. M. Lüscher, hep-lat/9909150.
4. F. Niedermayer, *Nucl. Phys. Proc. Suppl.* **73**, 105 (1999).
5. M. Lüscher, *Nucl. Phys. B* **549**, 295 (1999).
6. M. Lüscher, hep-lat/9904009.
7. P. Ginsparg, K. Wilson, *Phys. Rev. D* **25**, 2649 (1982).
8. H. Neuberger, *Phys. Lett. B* **417**, 141 (1998).
9. H. Neuberger, *Phys. Rev. D* **57**, 5417 (1998).
10. T. Kato, "Perturbation Theory for Linear Operators", Springer-Verlag, Berlin, 1984.
11. J. Avron, R. Seiler, B. Simon, *Commun. Math. Phys.* **159**, 399 (1994).
12. R. Narayanan, H. Neuberger, *Nucl. Phys. B* **443**, 305 (1995).
13. M. Testa, hep-lat/9707007.
14. H. Neuberger, *Nucl. Phys. Proc. Suppl.* **73**, 697 (1999).
15. R. Narayanan, H. Neuberger, *Nucl. Phys. B* **412**, 574 (1994).
16. S. Randjbar-Daemi, J. Strathdee, *Phys. Lett. B* **402**, 134 (1997).
17. H. Neuberger, *Phys. Rev. D* **59**, 085006 (1999).
18. W. Bardeen, B. Zumino, *Nucl. Phys. B* **244**, 421 (1984).
19. M. Berry, *Proc. R. Soc. London, Ser. A* **392**, 45 (1984).
20. R. Narayanan, H. Neuberger, *Phys. Rev. Lett.* **71**, 3251 (1993).
21. J. Avron, R. Seiler, B. Simon, *Phys. Rev. Lett.* **51**, 51 (1983).
22. L. O'RaiFeartaigh, "Group Structure of Gauge Theories", Cambridge University Press, 1986.
23. H. Neuberger, *Phys. Lett. B* **434**, 99 (1998).
24. E. Witten, *Phys. Lett. B* **117**, 324 (82).
25. H. Neuberger, *Phys. Lett. B* **437**, 117 (1998).
26. T. Kato, *J. Phys. Soc. Jpn.* **5**, 435 (1950).
27. B. Simon, *Phys. Rev. Lett.* **51**, 2167 (1983).
28. R. D. Ball, *Phys. Rept.* **182**, 1 (1989).